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REVIEW PAPER ON THE RUNGE-KUTTA METHODS TO STUDY NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The main purpose of this article is to review the work on Runge-Kutta Methods to study on Numerical Solutions of Initial Value Problems in Ordinary Differential Equations during the period 1983 to2020.The required material from 1983 to 1996pertaining to this research article has been taken freely from Hull et al. with gratitude and sincere acknowledgment. The status on this subject from 1983 to 2020 has been compiled by the present authors for the convenience of the new researchers entering in this field.

KEYWORDS: *Initial Value Problem, Numerical Solutions, Ordinary Differential Equations, Runge-Kutta Methods.*

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INTRODUCTION

A differential equation is an equation that relates one or several functions and its derivatives. Differential equations help us to understand phenomena that involves rate of change. These equations are the most important mathematical tools in generating models in Engineering, Biological and Physical Sciences. The differential equations are applied as mathematical representations of many actual world problems. For example, differential equations help us understand disease spread, weather and climate prediction, traffic flow, financial markets, population growth, water pollution, chemical reactions, suspension bridges, brain function, tumor growth, radioactive decay, electrical circuits, planetary motion and vibrations of guitar strings. An important type of problem that we must solve when we study ordinary differential equations is an initial value problem. An "Initial Value Problem" is an ordinary differential equation together with the conditions imposed on the unknown function and the values of its derivatives in a single number is called an initial value problem. At some point, the initial value problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. Numerical methods are generally used to solve mathematical problems that are formulated in science and engineering, where it is difficult to get an exact solution, only a few differential equations can be solved analytically. There are several analytical methods for solving ordinary differential equations. But there are a lot of ordinary differential equations that cannot be solved analytically. At that stage we have to use the numerical methods to solve this type of ordinary differential equations. There are several numerical methods for solving initial value problems. While studying literature, we came across the numerous works of numerical solution of Initial Value Problems applying the Runge-Kutta first, second,third

and fourth-order methods. Several authors have tried to solve initial value problems to obtain good precision using the methods mentioned above. Hull *et al*. [1996] completed the literature on the Runge-Kutta methods from 1957 to 1996 and the further update at till date, present authors have made an attempt.

Numerical Methods

Numerical method forms an important part of solving initial value problems in ordinary differential equations, most especially in cases where there is no closed form solution. Numerical methods are important to use to solve those differential equations whose exact solutions are not possible to obtain using analytic methods. Numerical methods are explicit or implicit, computed in one step or multiple steps. An explicit method computes the numerical solution at the next time point using the previous numerical solution at the previous time point. While an implicit method evaluates a function using the numerical solution at the next time point which is solved for. C. Runge and M.W. Kutta developed explicit and implicit Runge-Kutta methods [2015]. In 1895, C. Runge introduced Runge-Kutta first order and second order methods, K. Heun developed the third order Runge-Kutta method in [1996]. In 1901, W. Kutta introduced the fourth order and fifth order Runge-Kutta methods. In 1964, Butcher introduced the sixth order Runge-Kutta method. The seventh and eighth order methods are introduced by Curits [1996]. Runge-Kutta methods are active in research for the past several years. In several papers researchers have investigated and compared lower order and high order Runge-Kutta methods in terms of accuracy, stability and efficiency.

REVIEW OF LITERATURE FROM 1983 TO 2020

Maximum work in this period on Runge-Kutta methods has been done by Enright*et al.* [1994],Owren and Zennaro [1991,1992], Muir and Owren [1993], and Verner [1993].Enright *et al.*[1986] worked on Interpolants for Runge- Kutta formulas. Enright [1993] studied about the relative efficiency of alternative defect control schemes for high Runge-Kutta formulas have also been applied to singular initial-value problems by Enright and Suhartanto [1992].Peterson [1986] and Higham [1991] analysed global error using defect correction techniques for explicit Runge-Kutta methods. Detecting stiffness with explicit Runge-Kutta methods by Robertson [1986], some special Runge-Kutta formulas have been developed and implemented for boundary-value problems by Muir [1984] and Enright and Muir [1986]. Chan and Jackson [1986] used iterative linear equation solvers in codes for large systems of stiff initial value problems. Sharp [1987, 1989] worked on Runge-Kutta-Nystrom integrator for second order initial value problem and he analysed new low order explicit Runge-Kutta pairs respectively. Still others have been developed and implemented for first-order problems by Sharp [1989], and second-order problems by Sharp and Fine [1987,1992] Investigations related to the assessment and comparison of numerical methods have continued. Enright [1989, 1991] analysed error control strategies for continuous Runge-Kutta methods and a new error control for initial value problems respectively. Sharp [1991] has developed new formats for viewing and reporting the results of comparisons. The potential for parallelism in standard Runge-Kutta Methods has been studied by Jackson and Norsett [1995] both negative and positive results are presented. Many of thenegative results are based on a theorem that bounds the order of a Runge-Kutta formula in terms of the minimal polynomial associated with its coefficient matrix. The positive results are largely examples of prototypical formulas which offer a potential for effective "coarse-grain" parallelism on machines with a few processors. Runge-Kutta predictor-corrector methods have been discussed by Enenkel [1988] and analysed by Jackson, Kvarno and Norsett [1994]. Broderick *et.al*[1994]coupled mode equations with free carrier effects. Suhartan to [1990] a new approach detected singular points in the numerical solution of initial value problems. Nguyen [1995] worked on interpolation and error control schemes for algebraic differential equations using continuous implicit Runge-Kutta methods. Jackson [1991] did a survey of parallel numericalmethods for initial value problems for ODEs. Jackson and Pancer

[1992] worked on the parallel solution of ABD system arising in numerical methods for BVPs for ODEs. Enright and Macdonald [1992] worked on the practical implications of order reduction for implicit Runge-Kutta methods. Hayashi [1996] worked on numerical solution of retarded and neutral delay differential equations using continuous Runge-Kutta methods. Also Enright and Muir [1996] worked on a Runge-Kuttatype boundary value ODE solver with defect control. Enright *et al.* [1988] analysed effective solution of discontinuous IVPs using Runge-Kutta formulas pair with interpolants. Enright [1986,1996] analysed parallel defect control and convergence analysis for the numerical solution of retarded and neutral delay differential equations by continuous numerical methods. The results in the last two papers are of value not only for the development of parallel ODE codes, but also for the efficient, reliable and robust implementation of implicit Runge-Kutta methods on standard sequential machines.Onumanyi*et.al.*[1999]studied continuous finite difference approximations for solving differential equations". Gander [1999] gave a thoughtful consideration on the derivation of numerical methods using computer algebra. Hong[2000] worked on "The calculation of global error for initial value problem of ordinary differential equations". Ling*et al.* [2000] studied some application of Runge-Kutta–Merson Algorithm for creep damage analysis. Further, Boyce [2000] worked on elementary differential equationsand boundary value problems.Runge–Kutta with higher order derivative approximations was appeared in a paper by Goekenand Johnson [2000].Cockburn [2001] studied about Runge-Kutta discontinuous Galerkin Method for convections dominated problems.

Pimenov [2001] provides general linear methods for the numerical solution of functional-Differential Equations. Bernardo and Wang [2001] gave some aspects about Runge-Kutta discontinuous Galerkin Methods for convection-dominated problems. Awoyemi [2001] discovered a new sixth-order algorithm for general second order Ordinary Differential Equation. Fredebeul *et al.* [2002] worked on multiple order double output Runge-Kutta-Fehlberg formulae and some strategies for its efficient applications. Murugesan[2002] hasstudied a fourth order embedded Runge-Kutta Method based on arithmetic and centroidal means with error control.. Gerald [2002] worked upon applied numerical analysis. Berdan [2002], Butcher [2003, 2008] studied about numerical method for ordinary differential equations. Awoyemi [2003], studied P-stable linear multistep method for solving general third order ordinary differential equations. Also, Butcher [2003] gives detailed studies on numerical methods for ordinary differential equations. Biazar*et al.*[2004]obtained solution of the system of ordinary differential equations by Adomian decomposition method. Mathews [2005] gave many results in the book numerical methods for mathematics. Majid[2006]worked on higher order systems of ordinary differential equations block backward differentiation formula for solving first-order. Chapra [2006] analysed numerical method with the help of MATLAB. Ibrahim [2007] worked on "variable step block backward differentiation formula for solving first-order stiff ODEs". Akanvi [2010] analysed propagation of errors in Euler method. Atkinson [2009] gave numerical solutions of ordinary differential equation.

Hossain [2013, 2019] analysed a comparative solution of initial value problems by using modified Euler method and Runge-Kutta method. Bosede*et al.* [2012] gave some numerical methods in their famous paper entitled "on some numerical methods for solving initial value problems in ordinary differential equations". Also, Ogunrinde [2012] studied some numerical methods for solving initial value problems in ordinary differential equations. Rabiei [2012] improved the fifth-order Runge-Kutta method for solving ordinary differential equation. Islam [2015] analysed the accuracy of numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE). He also gave the accurate solutions of initial value problems for ordinary differential equations with the fourth order Runge-Kutta method. Islam [2015] studied in detail about numerical solutions of initial value problems. Further, Islam in [2015], performed a comparative study on numerical

solutions of initial value problems (IVP) for ordinary differential equations (ODE) with Euler and Runge-Kutta methods. Gowri *et al.* [2017]discussed about Runge-Kutta fourth order method with differential equations and its application. The differential equation problems have been solved by Runge-Kutta fourth order method and application problem are discussed with Runge-Kutta fourth order. He concluded that Runge-Kutta fourth order Method gave more accurate results. Sadiq *et al.* [2017] worked on using fourth order Ruge-Kutta Method to solve Lü Chaotic System. He observed that the accuracy of Runge-Kutta fourth order method solution can be increased by lessening the time step and shows that Runge-Kutta fourth order method successfully to solve theLü system. Hossain[2017] worked on a paper entitled "a study on the numerical solutions of second order initial value problems (IVP) for ordinary differential equations with fourth order and Butcher's fifth order Runge-Kutta methods". Hamed, [2017] gave the accuracy of Euler and modified Euler technique for first order ordinary differential equations with initial conditions. Samsudin *et al.*[2018]worked on cube arithmetic: improving Euler method for ordinary differential equation analysis and comparative study of numerical solutions of initial value problems (IVP) in ordinary differential equations with Euler and Runge-Kutta methods. Anthony [2019]*et al.* analysed and compared the numerical solutions of initial value problems. They compared the performance and the computational effort of the two methods and suggested that to achieve more accuracy in the solutions, the step size needs to be very small. Jamali [2019] produced the analysis and the comparative study of numerical methods to solve ordinary differential equations with initial value problems. Murad Hossen *et al.* (2019) worked on a comparative investigation on numerical solution of initial value problem by using modified Euler's method and Runge-Kutta method. In this article, Modified Euler's method and Runge-Kutta methods have been used to find the numerical solutions of ordinary differential equations with initial value problems. By using MATLAB. Mohammad *et al.*[2020] worked on a comparative exploration on different numerical methods for solving ordinary differential equations.

CONCLUSIONS

We have tried to go through each and every research article on this subject till date. It is very clear from this compilation that majority of the work on this topic comes from the workers overseas and they have dealt mainly with the accuracy and comparative analysis on Runge-Kutta first, second, third, and fourth order method. This review reflects that there is hardly any report of work available in the Indian scenario. Hence there is a strong need of taking up detailed studies on this problems which will prove beneficial in solving the initial value problems.

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